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Nondifferentiable multiobjective higher-order duality relations for unified type dual models under type-I functions

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Abstract

The motivation behind this paper is to study a class of nondifferentiable multiobjective fractional programming problem in which each component of objective functions contains a term including the support function of a compact convex set. For a differentiable function, we consider the class of second-order (F, α, ρ, d)-type-I convex functions. Further, we formulate unified (mixed type) dual models and derived duality relations under aforesaid assumptions.

Keywords: Duality, support function, type-I functions, higher order pseudo quasi (F, α, ρ, d)-type I, unified dual, efficient solutions
Mathematics Subject Classification 90C26 . 90C30 . 90C32 . 90C46

1. Introduction

Higher-order duality in nonlinear programming has been studied by many researchers. One computational advantage of higher-order duality over first order duality is that it provides tighter bounds for the value of the objective function of the primal problem when approximations are used due to more parameters. In various numerical algorithms, higher-order duality is considered over first-order as it gives more closer bounds. Duality relations for multiobjective programming problems with generalized convexity assumptions were given by several researchers.

Mangasarian [7] considered a nonlinear program and discussed second order duality under certain inequalities. The concept of higher-order convexity presented by Zhang [10] and derived duality results in multiobjective programming problem. Later on, Yang et al. [9] considered a unified higher-order dual model for nondifferentiable multiobjective programs and proved duality results under generalized assumptions. Suneja et al. [8] introduced a higher order (F, α, σ)-type I functions and formulated higher order dual programs for a nondifferentiable multiobjective fractional programming problem.

Motivated by various concepts of generalized convexity, Ahmad [1] formulated unified higher order dual for a nondifferentiable multiobjective programming problem, where every component of the objective function contains a term involving the support function of a compact convex set and established duality results. Further, Gulati and Agarwal [3] gave the concept of second-order -V-type I functions for multiobjective programming problem which were recently extended to nondifferentiable case by Jayswal et al. [5]. Recently,

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Jayswal et al. [6] considerded higher-Order duality for multiobjective programming problems and discussed duality theorems under $(F, \alpha, \rho, d) - V-$ type-I functions.

In this paper, we have generalized the definition of higher-order (F, α, ρ, d)-type-I functions for a nondifferentiable multiobjective higher -order fractional programming problem. We have formulated higher-order unified dual and established duality results under (F, α, ρ, d)-type-I assumptions. By using lemma, we have formulated generalized unified dual model and proved duality results under generalized assumptions.

2. Preliminaries and Definitions

Definition 2.1. Let $Q \subseteq R^n$ be a compact convex set. The support function of Q is defined by

$$s\,(y|Q) = \max\{y^T z : z \in Q\}.$$

Consider the following nondifferentiable multiobjective fractional programming problem:

$$\textbf{(MFP)}\ \text{Minimize}\ \Psi(x) = \left(\frac{\phi_1(x) + s(x|C_1)}{\psi_1(x) - s(x|E_1)}, \frac{\phi_2(x) + s(x|C_2)}{\psi_2(x) - s(x|E_2)}, ..., \frac{\phi_k(x) + s(x|C_k)}{\psi_k(x) - s(x|E_k)}\right)^T$$
$$\text{subject to } x \in Y^0 = \{x \in Y : \pi_j(x) + s(x|D_j) \leq 0, j \in M\},$$

where $Y \subseteq R^n$ is an open set. The functions $\phi,\ \psi : Y \to R^k,\ \pi : Y \to R^m$ are differentiable on Y and $C_i,\ E_i,\ D_j$ are compact convex sets in R^n for $i \in K$ and $j \in M$. Let $\phi_i(x) + s(x|C_i) \geq 0$ and $\psi_i(x) - s(x|E_i) > 0,\ i \in K$. Let $\alpha \in R_+ \setminus \{0\},\ \rho \in R$ and $d(x,z) = 0 \Leftrightarrow x = z$.

Definition 2.2[6]. A functional $F : Y \times Y \times R^n \to R$ is sublinear with its third position if $\forall\ (x,u) \in Y \times Y$,

(i) $F_{x,u}(a_1 + a_2) \leqq F_{x,u}(a_1) + F_{x,u}(a_2),\ \forall\ a_1,\ a_2 \in R^n$,
(ii) $F_{x,u}(\alpha a) = \alpha F_{x,u}(a),\ \forall\ \alpha \in R_+$ and $a \in R^n$.

Definition 2.3[2]. $u \in Y^0$ is efficient solution of (MFP) if $\exists$ no $x \in Y^0$ such that $\Psi(x) \leq \Psi(u)$.

Definition 2.4. $\left(\frac{\phi_i(.) + (.)^T z_i}{\psi_i(.) - (.)^T v_i},\quad \pi_j(.) + (.)^T w_j\right)$ is higher order pseudo quasi (F, α, ρ, d)-type I at u of (MFP) if $\forall\ x \in Y^0$ and $p \in R^n$

$$\frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} < \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u,p) - p^T \nabla_p H_i(u,p) \Rightarrow F\left(x, u, \alpha^1(x,u)\left\{\nabla\left(\frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i}\right) + \nabla_p H_i(u,p)\right) + \rho^1 d$$

and

$$-\pi_j(u) - u^T w_j \leq K_j(u,p) - p^T \nabla_p K_j(u,p) \Rightarrow F(x, u, \alpha^2(x,u)\{\nabla \pi_j(u) + \nabla_p K_j(u,p)\}) + \rho^2 d^2(x,u) \leq 0.$$

Definition 2.5. $\left(\frac{\phi_i(.) + (.)^T z_i}{\psi_i(.) - (.)^T v_i},\quad \pi_j(.) + (.)^T w_j\right)$ is higher order strictly pseudo quasi (F, α, ρ, d)-type I at u of (MFP) if $\forall\ x \in Y^0$ and $p \in R^n$

$$\frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} \leq \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u,p) - p^T \nabla_p H_i(u,p) \Rightarrow F\left(x, u, \alpha^1(x,u)\left\{\nabla\left(\frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i}\right) + \nabla_p H_i(u,p)\right) + \rho^1 d$$

and

$$-\pi_j(u) - u^T w_j \leq K_j(u,p) - p^T \nabla_p K_j(u,p) \Rightarrow F(x, u, \alpha^2(x,u)\{\nabla \pi_j(u) + \nabla_p K_j(u,p)) + \rho^2 d^2(x,u) \leq 0.$$

Definition 2.6. $\left(\dfrac{\phi_i(.)+(.)^T z_i}{\psi_i(.)-(.)^T v_i}, \pi_j(.)+(.)^T w_j\right)$ is higher order weak strictly pseudo quasi (F,α,ρ,d)-type I at u of (MFP) if $\forall\ x \in Y^0$ and $p \in R^n$

$$\frac{\phi_i(x)+x^T z_i}{\psi_i(x)-x^T v_i} \le \frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i} + H_i(u,p) - p^T\nabla_p H_i(u,p) \Rightarrow F\left(x,u,\alpha^1(x,u)\left\{\nabla\left(\frac{\phi_i(u)+u^T z_i}{\psi_i(u)-u^T v_i}\right)+\nabla_p H_i(u,p)\right\}\right)+\rho$$

and

$$-\pi_j(u)-u^T w_j \le K_j(u,p)-p^T\nabla_p K_j(u,p) \Rightarrow F(x,u,\alpha^2(x,u)\{\nabla\pi_j(u)+\nabla_p K_j(u,p)\})+\rho^2 d^2(x,u)\le 0.$$

Theorem 2.1 (K-K-T-type necessary condition)[4]. Let u be efficient solution of (MFP) at which the Kuhn-Tucker constraint qualification is satisfied on X. Then, $\exists\ 0<\bar{\lambda}\in R^k,\ 0\le \bar{y}_j\in R^m,\ \bar{z}_i\in R^n,\ \bar{v}_i,\ \bar{w}_j\in R^n,\ i\in K,\ j\in M$ such that

$$\sum_{i=1}^{k}\bar{\lambda}_i\nabla\left(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\right)+\sum_{j=1}^{m}\bar{y}_j\nabla(\pi_j(u)+u^T\bar{w}_j)=0, \tag{1}$$

$$\sum_{j=1}^{m}\bar{y}_j(\pi_j(u)+u^T\bar{w}_j)=0, \tag{2}$$

$$u^T\bar{z}_i=S(u|C_i),\ u^T\bar{v}_i=S(u|E_i),\ u^T\bar{w}_j=S(u|D_j), \tag{3}$$

$$\bar{z}_i\in C_i,\ \bar{v}_i\in D_i,\ \bar{w}_j\in E_j,\ i\in K,\ j\in M. \tag{4}$$

3. Unified higher-order duality model-I:

In this section, we formulate the following unified higher- order dual for (MFP) and establish duality theorems:

(HMDP): Maximize $\Bigg(\dfrac{\phi_1(y)+y^T z_1}{\psi_1(y)-y^T v_1}+H_1(y,p)-p^T\nabla_p H_1(y,p)+\sum_{j\in J_0}\mu_j\{\pi_j(y)+y^T w_j+K_j(y,p)-p^T\nabla_p K_j(y,p)\}$

$$,\ldots,\frac{\phi_k(y)+y^T z_k}{\psi_k(y)-y^T v_k}+H_k(y,p)-p^T\nabla_p H_k(y,p)+\sum_{j\in J_0}\mu_j\{\pi_j(y)+y^T w_j+K_j(y,p)-p^T\nabla_p K_j(y,p)\}\Bigg)$$

subject to $y\in Y$,

$$\sum_{i=1}^{k}\lambda_i\left\{\nabla\left(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\right)+\nabla_p H_i(y,p)\right\}+\sum_{j=1}^{m}\mu_j\{\nabla\pi_j(y)+w_j+\nabla_p K_j(y,p)\}=0, \tag{5}$$

$$\sum_{j\in J_\beta}\mu_j\{\pi_j(y)+y^T w_j+K_j(y,p)-p^T\nabla_p K_j(y,p)\}\ge 0,\ \beta=1,\ldots,r, \tag{6}$$

$$\lambda_i \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1, \tag{7}$$

$$\mu_j \geq 0, \ z_i \in C_i, \ v_i \in E_i, \ w_j \in D_j \ \text{ for } i \in K, \ j \in M, \tag{8}$$

where $J_\delta \subseteq N$, $\delta = 0, 1, ..., r$ with $\bigcup_{\delta=0}^{r} J_\delta = N$ and $J_{\delta_1} \bigcap J_{\delta_2}$ if $\delta_1 \neq \delta_2$. It may be noted that $J_0 = N$ and $J_\beta = \phi(1 \leq \beta \leq r)$, we obtain Wolfe type dual. If $J_0 = \phi$, $J_1 = N$ and $J_\beta = \phi \ \ (2 \leq \beta \leq r)$, then (HMDP) reduces to Mond-Weir Type dual.

Let W^0 be feasible solution of (HMDP).

Theorem 3.1 (Weak Duality Theorem). Let $x \in Y^0$ and $(y, \lambda, v, \mu, z, w, p) \in W^0$. Let

(i) $\left(\dfrac{\phi_i(.) + (.)^T z_i}{\psi_i(.) - (.)^T v_i} + \mu_{J_0}^T \left(\pi_j J_0 + (.)^T w_j J_0 \right) e, \{\pi_j(.) + (.)^T w_j\}_{J_\beta}^{\mu} \right)$ be higher- order weak strictly pseudo quasi (F, α, ρ, d)-type I at y,

(ii) $\displaystyle\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1}{\alpha^1(x,y)} + \sum_{j=1}^{r} \frac{\rho_j^2}{\alpha^2(x,y)} \geq 0.$

Then, the following cannot hold

$$\frac{\phi_i(x) + s(x|C_i)}{\psi_i(x) - s(x|E_i)} \leq \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} + H_i(y,p) - p^T \nabla_p H_i(y,p) + \sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y,p) - p^T \nabla_p K_j(y,p)\}, \forall\ i \in$$

and

$$\frac{\phi_j(x) + s(x|C_j)}{\psi_j(x) - s(x|E_j)} < \frac{\phi_j(y) + y^T z_j}{\psi_j(x) - y^T v_j} + H_j(y,p) - p^T \nabla_p H_j(y,p) + \sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y,p) - p^T \nabla_p K_j(y,p)\}, \text{for s}$$

Proof If possible, then suppose inequalities (9) and (10) hold. As $x^T z_i \leq s(x|C_i)$, $x^T v_i \leq s(x|E_i)$, $\forall i \in K$ and $\displaystyle\sum_{j \in J_0} \mu_j (\pi_j(x) + x^T w_j) \leq 0$, using the inequalities and the dual constraint (6), hypothesis (i) gives

$$F\bigg(x, y, \alpha^1(x,y) \bigg\{ \nabla \bigg(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \bigg) + \nabla_p H_i(y,p) + \sum_{j \in J_0} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\} e \bigg\} \bigg) < -\rho^1 d^2(x,y)$$

and

$$F(x, y, \alpha^2(x,y) \sum_{j \in J_\beta} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\}) + \rho_\beta^2 d^2(x,y) \leq 0, \ \beta = 1, ..., r.$$

Since $\lambda \geq 0$, $\lambda^T e = 1$, it follows that

$$F\bigg(x, y, \sum_{i=1}^{k} \lambda_i \bigg\{ \nabla \bigg(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \bigg) + \nabla_p H_i(y,p) \bigg\} + \sum_{j \in J_0} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\} \bigg) < -\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d^2(x,y)}{\alpha^1(x,y)}$$

and

$$F\bigg(x, y, \sum_{j \in J_\beta} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\} \bigg) \leq -\frac{\rho_\beta^2 d^2(x,y)}{\alpha^2(x,y)}, \ \beta = 1, ..., r.$$

Using, the concept of sublinearity of F, we have

$$F\Bigg(x, y, \sum_{i=1}^{k} \lambda_i \Bigg\{ \nabla\Bigg(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Bigg) + \nabla_p H_i(y,p) \Bigg\} + \sum_{j=1}^{m} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\}\Bigg)$$

$$= F\Bigg(x, y, \sum_{i=1}^{k} \lambda_i \Bigg\{ \nabla\Bigg(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Bigg) + \nabla_p H_i(y,p) \Bigg\} + \sum_{j\in J_0} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\} + \sum_{j\in J_1} \mu_j \{\nabla \pi_j(y) + w_j + \nabla$$

$$+ ... + \sum_{j\in J_r} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p$$

$$F\Bigg(x, y, \sum_{i=1}^{k} \lambda_i \Bigg\{ \nabla\Bigg(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Bigg) + \nabla_p H_i(y,p) \Bigg\} + \sum_{j=1}^{m} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\}\Bigg)$$

$$\le F\Bigg(x, y, \sum_{i=1}^{k} \lambda_i \Bigg\{ \nabla\Bigg(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Bigg) + \nabla_p H_i(y,p) \Bigg\} + \sum_{j\in J_0} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\}\Bigg)$$

$$+ F\Bigg(x, y, \sum_{j\in J_1} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\}\Bigg) + ... + F\Bigg(x, y, \sum_{j\in J_r} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,$$

$$< -\Bigg(\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1}{\alpha^1(x,y)} + \sum_{j=1}^{r} \frac{\rho_j^2}{\alpha^2(x,y)}\Bigg) d^2(x,y$$

Further, using hypothesis (ii), we have

$$F\Bigg(x, y, \sum_{i=1}^{k} \lambda_i \Bigg\{ \nabla\Bigg(\frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i}\Bigg) + \nabla_p H_i(y,p)\Bigg) \Bigg\} + \sum_{j=1}^{m} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y,p)\}\Bigg) < 0,$$

which contradicts (5). Hence, completes the proof.

Theorem 3.2 (Weak Duality Theorem). Let $x \in Y^0$ and $(y, \lambda, v, \mu, z, w, p) \in W^0$. Let
(i) $\left(\frac{\phi_i(.)+(.)^T z_i}{\psi_i(.)+(.)^T v_i} + \mu_{J_0}^T \left(\pi_j J_0 + (.)^T w_j J_0\right) e, \{\pi_j(.) + (.)^T w_j\}_{J_\beta}^{\mu}\right)$ be higher- order pseudo quasi (F, α, ρ, d)-type I at y,
(ii) $\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1}{\alpha^1(x,y)} + \sum_{j=1}^{r} \frac{\rho_j^2}{\alpha^2(x,y)} \ge 0.$
Then, the following cannot hold

$$\frac{\phi_i(x)+s(x|C_i)}{\psi_i(x)-s(x|E_i)} \le \frac{\phi_i(y)+y^T z_i}{\psi_i(y)-y^T v_i} + H_i(y,p) - p^T \nabla_p H_i(y,p) + \sum_{j\in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y,p) - p^T \nabla_p K_j(y,p)\},\ \forall\ i \in$$

and

$$\frac{\phi_j(x)+s(x|C_j)}{\psi_j(x)-s(x|E_j)} < \frac{\phi_j(y)+y^T z_j}{\psi_j(x)-y^T v_j} + H_j(y,p) - p^T \nabla_p H_j(y,p) + \sum_{j\in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y,p) - p^T \nabla_p K_j(y,p)\}, \text{for s}$$

Proof The proof follows on the lines of Theorem 3.1.

Theorem 3.3 (Strong Duality Theorem). If $\bar{u}$ is an efficient solution of (MFP) and let the Kuhn-Tucker constraint qualification be satisfied. Then, $\exists\, \bar{\lambda} \in R^k,\ \bar{y} \in R^m,\ \bar{z_i} \in R^n,\ \bar{v_i} \in R^n$ and $\bar{w_j} \in R^n,\ i \in K,\ j \in M,$ such that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p}) \in W^0$ and the (MFP)and (HMDP) have equal values. Also, if

$$H(\bar{u}, 0) = 0,\ \nabla_p H(\bar{u}, 0) = 0,\ K(\bar{u}, 0) = 0,\ \nabla_p K(\bar{u}, 0) = 0.$$

Furthermore, if the assumptions of Theorem 3.1 hold for Y^0 and W^0, then $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is an efficient solution of (HMDP).

Proof. From the given conditions in the statement and using Theorem 2.1, we get the proof of Strong duality Theorem.

□

Theorem 3.4 **(Strict Converse Duality Theorem)**. Let $u \in Y^0$ and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{v}, \bar{z}, \bar{w}, \bar{p}) \in W^0$ such that

(i) $$\sum_{i=1}^{k} \bar{\lambda_i} \left\{ \frac{\phi_i(u) + u^T \bar{z_i}}{\psi_i(u) - u^T \bar{v_i}} \right\} \leq \sum_{i=1}^{k} \bar{\lambda_i} \left\{ \frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z_i}}{\psi_i(\bar{y}) - \bar{y}^T \bar{v_i}} + H_i(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p H_i(\bar{y}, \bar{p}) + \sum_{j \in J_0} \bar{\mu_j} \{\pi_j(\bar{y}) + \bar{y}^T \bar{w_j} + K_j(\bar{y}, \bar{p})\right.$$

(ii) $$\frac{\rho_i^1}{\alpha^1(u, \bar{y})} + \sum_{j=1}^{r} \frac{\rho_j^2}{\alpha^2(u, \bar{y})} \geq 0,\ \forall\ i,\ j,$$

(iii) $$\left(\sum_{i=1}^{k} \bar{\lambda_i} \left\{ \frac{\phi_i(.) + (.)^T \bar{z_i}}{\psi_i(.) - (.)^T \bar{v_i}} \right\} + \sum_{j \in J_0} \bar{\mu_j}\ \{\pi_j + (.)^T \bar{w_j}\}, \{\pi_j(.) + (.)^T \bar{w_j}\}_{J_\beta}^{\bar{\mu}} \right)$$ is higher order strictly pseudoquasi(F,

Then, $u = \bar{y}$.

Proof Suppose $u \neq \bar{y}$. The dual constraint (6) and the hypothesis (iii), for $\beta = 1, ..., r$ yield

$$F(u, \bar{y}, \alpha^2(u, \bar{y}) \sum_{j \in J_\beta} \bar{\mu_j} \{\nabla \pi_j(\bar{y}) + \nabla^2 \pi_j(\bar{y}) p + \bar{w_j}\}) \leq -\rho_\beta^2 d^2(u, \bar{y}) \tag{13}$$

By the dual constraint (5), we have

$$F\left(u, \bar{y}, \sum_{i=1}^{k} \bar{\lambda_i} \left\{ \nabla \left(\frac{\phi_i(u) + u^T \bar{z_i}}{\psi_i(u) - u^T \bar{v_i}} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right\} + \sum_{j=1}^{m} \bar{\mu_j} \left\{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w_j} \right\} \right) = 0$$

which by sublinearity of F alongwith 13 give

$$F\left(u, \bar{y}, \sum_{i=1}^{k} \bar{\lambda_i} \left\{ \nabla \left(\frac{\phi_i(u) + u^T \bar{z_i}}{\psi_i(u) - u^T \bar{v_i}} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right) \right\} + \sum_{j \in J_0} \bar{\mu_j} \left\{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w_j} \right\} \right)$$

$$\geq -F\left(u, \bar{y}, \sum_{j \in J_1} \bar{\mu_j} \{\nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w_j}\}\right) - ... - F(u, \bar{y}, \sum_{j \in J_r} \bar{\mu_j} \{\nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w_j}\} \Big)$$

or

$$F\left(u, \bar{y}, \sum_{i=1}^{k} \bar{\lambda_i} \left\{ \nabla \left(\frac{\phi_i(u) + u^T \bar{z_i}}{\psi_i(u) - u^T \bar{v_i}} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right) \right\} + \sum_{j \in J_0} \bar{\mu_j} \left\{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w_j} \right\} \right) \geq \sum_{j=1}^{r} \frac{\rho_j^2 d^2(u, \bar{y})}{\alpha^2(u, \bar{y})},$$

by hypothesis (ii),

$$F\left(u, \bar{y}, \sum_{i=1}^{k} \bar{\lambda_i} \left\{ \nabla \left(\frac{\phi_i(u) + u^T \bar{z_i}}{\psi_i(u) - u^T \bar{v_i}} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right\} + \sum_{j \in J_0} \bar{\mu_j} \left\{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w_j} \right\} \right) \geq \frac{-\rho_i^1 d^2(u, \bar{y})}{\alpha^1(u, \bar{y})}.$$

It follows that

$$F\left(u,\bar{y},\sum_{i=1}^{k}\bar{\lambda}_i\left\{\nabla\left(\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\right)+\nabla_p H_i(\bar{y},\bar{p})\right\}+\sum_{j\in J_0}\bar{\mu}_j\left\{\nabla\pi_j(\bar{y})+\nabla_p K_j(\bar{y},\bar{p})+\bar{w}_j\right\}\right)\geq\frac{-\rho_i^1 d^2(u,\bar{y})}{\alpha^1(u,\bar{y})}.$$

Therefore hypothesis (iii) in view of $\sum_{j\in J_0}\mu_j\{\pi_j(u)+u^T\bar{w}_j\}\leq 0$ yields

$$\sum_{i=1}^{k}\bar{\lambda}_i\left\{\frac{\phi_i(u)+u^T\bar{z}_i}{\psi_i(u)-u^T\bar{v}_i}\right\}>\sum_{i=1}^{k}\bar{\lambda}_i\left\{\frac{\phi_i(\bar{y})+\bar{y}^T\bar{z}_i}{\psi_i(\bar{y})-\bar{y}^T\bar{v}_i}+H_i(\bar{y},\bar{p})-\bar{p}^T\nabla_p H_i(\bar{y},\bar{p})\right\}+\sum_{j\in J_0}\bar{\mu}_j\{\pi_j(\bar{y})+\bar{y}^T\bar{w}_j+K_j(\bar{y},\bar{p})-\bar{p}^T\nabla_p$$

which contradicts hypothesis (i). Hence, the result.

For proving various duality results with the help of the given lemma:

Lemma 3.1 $u\in X$ is an efficient solution for (MFP) if and only if there exists $\bar{\nu}_i\in R_+^k$ such that u is an efficient solution for $(\text{MFP})_{\bar{\nu}_i}$, where $\bar{\nu}_i=\dfrac{\phi_i(u)+S(u|C_i)}{\psi_i(u)-S(u|E_i)},\ i=1,2,...,k.$

Next we formulate equivalent nondifferentiable multiobjective programming problem (MFP) with the help of lemma given as:

$$\textbf{(MFP)}_{\bar{\nu}_i}\ \text{Minimize}\ \pi(x)=\Big(\phi_1(x)+s(x|C_1)-\nu_1(\psi_1(x)-s(x|E_1)),\ \phi_2(x)+s(x|C_2)-\nu_2(\psi_2(x)-s(x|E_2))$$
$$,...,\phi_k(x)+s(x|C_k)-\nu_k(\psi_k(x)-s(x|E_k))\Big)^T$$

subject to $x\in Y^0=\{x\in Y:\pi_j(x)+s(x|D_j)\leq 0, j\in M\}$.

4. Unified higher-order duality model-II:

In this section, we formulate the following unified higher-order dual for $(\text{MFP})_{\bar{\nu}}$ and establish duality theorems:

$$\textbf{(HMDP)}_{\bar{\nu}_i}\textbf{:}\ \text{Maximize}\ \Big(\phi_1(y)+y^Tz_1-\nu_1(\psi_1(y)-y^Tv_1)+H_1(y,p)-p^T\nabla_p H_1(y,p)+\sum_{j\in J_0}\mu_j\{\pi_j(y)$$
$$+y^Tw_j+K_j(y,p)-p^T\nabla_p K_j(y,p),...,\phi_k(y)+y^Tz_k-\nu_k(\psi_k(y)-y^Tv_k)+K_j(y,p)-$$
$$p^T\nabla_p K_j(y,p)$$
$$+\textstyle\sum_{j\in J_0}\mu_j\{\pi_j(y)+y^Tw_j+K_j(y,p)-p^T\nabla_p K_j(y,p)\}\Big)$$

subject to $y\in Y$,

$$\sum_{i=1}^{k}\lambda_i\left\{\nabla(\phi_i(y)+y^Tz_i-\nu_i(\psi_i(y)-y^Tv_i))+\nabla_p H_i(y,p)\right\}+\sum_{j=1}^{m}\mu_j\{\nabla\pi_j(y)+w_j+\nabla_p K_j(y,p)\}=0,\quad(14)$$

$$\sum_{j\in J_\beta}\mu_j\{\pi_j(y)+y^Tw_j+K_j(u,p)-p^T\nabla_p K_j(u,p)\}\geq 0,\ \beta=1,...,r,\quad(15)$$

$$\phi_i(y)+y^Tz_i-\nu_i(\psi_i(y)-y^Tv_i)\geq 0,\ \forall\ i,\quad(16)$$

$$\lambda_i \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1, \tag{17}$$

$$\mu_j \geq 0, \; z_i \in C_i, \; v_i \in E_i, \; w_j \in D_j \;\; \text{for } i \in K, \; j \in M, \tag{18}$$

where $J_\delta \subseteq N$, $\delta = 0, 1, ..., r$ with $\bigcup_{\delta=0}^{r} J_\delta = N$ and $J_{\delta_1} \bigcap J_{\delta_2}$ if $\delta_1 \neq \delta_2$. It may be noted that $J_0 = N$ and $J_\beta = \phi(1 \leq \beta \leq r)$, we obtain Wolfe type dual. If $J_0 = \phi$, $J_1 = N$ and $J_\beta = \phi(2 \leq \beta \leq r)$, then $(\text{HMDP})_{\bar{\nu}}$ reduces to Mond-Weir Type dual.

Let T^0 be feasible solution of $(\text{HMDP})_{\bar{\nu}}$.

Theorem 4.1 (Weak Duality Theorem). Let $xinY^0$ and $(y, \lambda, v, \mu, z, w, p) \in T^0$. Let

(i) $\left(\phi_i(.) + (.)^T z_i - \nu_i(\psi_i(.) + (.)^T v_i) + \mu_{J_0}^T \left(\pi_j J_0 + (.)^T w_j J_0\right) e, \; \{\pi_j(.) + (.)^T w_j\}_{J_\beta}^{\mu}\right)$ be higher- order weak strictly pseudo quasi (F, α, ρ, d)-type I at y,

(ii) $\displaystyle\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1}{\alpha^1(x, y)} + \sum_{j=1}^{r} \frac{\rho_j^2}{\alpha^2(x, y)} \geq 0.$

Then, the following cannot hold

$$\phi_i(x) + s(x|C_i) - \nu_i(\psi_i(x) - s(x|E_i)) \leq \phi_i(y) + y^T z_i - \nu_i(\psi_i(y) - y^T v_i) + H_i(y, p) - p^T \nabla_p H_i(y, p)$$
$$+ \sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p)\}, \; \forall \; i \in K \; (1$$

and

$$\phi_j(x) + s(x|C_j) - \nu_j(\psi_j(x) - s(x|E_j)) < \phi_j(y) + y^T z_j - \nu_j(\psi_j(x) - y^T v_j) + H_j(y, p) - p^T \nabla_p H_j(y, p)$$
$$+ \sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p)\}, \text{for at least one } j \in$$

Proof. The proof follows on the lines of Theorem 3.1. □

Theorem 4.2 (Weak Duality Theorem). Let $x \in Y^0$ and $(y, \lambda, v, \mu, z, w, p) \in T^0$. Let

(i) $\left(\phi_i(.) + (.)^T z_i - \nu_i(\psi_i(.) + (.)^T v_i) + \mu_{J_0}^T \left(\pi_j J_0 + (.)^T w_j J_0\right) e, \{\pi_j(.) + (.)^T w_j\}_{J_\beta}^{\mu}\right)$ be higher- order pseudo quasi (F, α, ρ, d)-type I at y,

(ii) $\displaystyle\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1}{\alpha^1(x, y)} + \sum_{j=1}^{r} \frac{\rho_j^2}{\alpha^2(x, y)} \geq 0.$

Then, the following cannot hold

$$\phi_i(x) + s(x|C_i) - \nu_i(\psi_i(x) - s(x|E_i)) \leq \phi_i(y) + y^T z_i - \nu_i(\psi_i(y) - y^T v_i) + H_i(y, p) - p^T \nabla_p H_i(y, p)$$
$$+ \sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p)\}, \quad \forall \; i \in K \; ($$

and

$$\phi_j(x) + s(x|C_j) - \nu_j(\psi_j(x) - s(x|E_j)) < \phi_j(y) + y^T z_j - \nu_j(\psi_j(x) - y^T v_j) + H_j(y, p) - p^T \nabla_p H_j(y, p)$$
$$+ \sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p)\}, \text{for some } j$$

Proof. The proof follows on the lines of Theorem 3.1. □

Theorem 4.3 (Strong Duality Theorem). Let $\bar{u}$ be efficient solution of $(\text{MFP})_{\bar{\nu}_i}$ and let the Kuhn-Tucker constraint qualification be satisfied. Then there exist $\bar{\lambda} \in R^k,\ \bar{y} \in R^m,\ \bar{z}_i \in R^n,\ \bar{v}_i \in R^n$ and $\bar{w}_j \in R^n,\ i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m$, such that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ is a feasible solution of $(\text{HMDP})_{\bar{\nu}}$ and the $(\text{MFP})_{\bar{\nu}}$ and $(\text{HMDP})_{\bar{\nu}}$ have equal values. Also, if

$$H(\bar{u}, 0) = 0,\ \nabla_p H(\bar{u}, 0) = 0,\ K(\bar{u}, 0) = 0,\ \nabla_p K(\bar{u}, 0) = 0.$$

Furthermore, if the assumptions of Theorem 4.1 hold for all Y^0 and $(\text{HMDP})_{\bar{\nu}}$, then $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is an efficient solution of $(\text{HMDP})_{\bar{\nu}}$.

Proof. From the given conditions in the statement and using Theorem 2.1, since Equation (1) can be written as,

$$\sum_{i=1}^{k} \frac{\bar{\mu}_i}{\psi_i(\bar{u}) - \bar{u}^T \bar{v}_i} \left(\nabla(\phi_i(\bar{u}) + \bar{u}^T \bar{z}_i) - \frac{\phi_i(\bar{u}) + \bar{u}^T \bar{z}_i}{\psi_i(\bar{u}) - \bar{u}^T \bar{v}_i} \nabla(\psi_i(\bar{u}) - \bar{u}^T \bar{v}_i) \right) + \sum_{j=1}^{m} \nabla \bar{y}_j (\pi_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0. \quad (23)$$

Letting $\bar{\lambda}_i = \dfrac{\bar{\mu}_i}{\psi_i(\bar{u}) - \bar{u}^T \bar{v}_i}$ and $\bar{\nu}_i = \dfrac{\phi_i(\bar{u}) + \bar{u}^T \bar{z}_i}{\psi_i(\bar{u}) - \bar{u}^T \bar{v}_i}$, $\quad i = 1, 2, ..., k$, we have

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla \left(\phi_i(\bar{u}) + \bar{u}^T \bar{z}_i - \bar{\nu}_i(\psi_i(\bar{u}) - \bar{u}^T \bar{v}_i) \right) + \sum_{j=1}^{m} \bar{y}_j \nabla(\pi_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0 \quad (24)$$

$$\phi_i(\bar{u}) + \bar{u}^T \bar{z}_i - \bar{\nu}_i(\psi_i(\bar{u}) - \bar{u}^T \bar{v}_i) = 0. \quad (25)$$

Thus, $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu}_i, \bar{w}, \bar{p} = 0)$ is feasible for $(\text{HMDP})_{\bar{\nu}}$ and the objective function values of $(\text{MFP})_{\bar{\nu}}$ and $(\text{HMDP})_{\bar{\nu}}$ are equal. Hence, completes the result. □

Theorem 4.4 **(Strict Converse Duality Theorem)**. Let $u \in Y^0$ and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{v}, \bar{z}, \bar{w}, \bar{p}) \in T^0$ such that

(i) $$\sum_{i=1}^{k} \bar{\lambda}_i \left\{ \phi_i(u) + u^T \bar{z}_i - \nu_i(\psi_i(u) - u^T \bar{v}_i) \right\} \leq \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \phi_i(u) + u^T \bar{z}_i - \nu_i(\psi_i(u) - u^T \bar{v}_i) + H_i(u, \bar{p}) - p^T \nabla_p H_i(u,\right.$$
$$+ \sum_{j \in J_0} \bar{\mu}_j \{ \pi_j(\bar{y}) + \bar{y}^T \bar{w}_j + K_j(u, \bar{p}) - p^T \nabla_p K_J(u, \bar{p}) \},$$

(ii) $$\frac{\rho_i^1}{\alpha^1(u, \bar{y})} + \sum_{j=1}^{r} \frac{\rho_j^2}{\alpha^2(u, \bar{y})} \geq 0,$$

(iii) $$\left(\sum_{i=1}^{k} \bar{\lambda}_i \left\{ \phi_i(u) + u^T \bar{z}_i - \nu_i(\psi_i(u) - u^T \bar{v}_i) \right\} + \sum_{j \in J_0} \bar{\mu}_j \ \{ \pi_j(.) + (.)^T \bar{w}_j \}, \{ \pi_j(.) + (.)^T \bar{w}_j \}_{J_\beta}^{\bar{\mu}} \right)$$ is higher- order st
(F, α, ρ, d) − type I at $\bar{y}$.

Then, $u = \bar{y}$.

Proof. The proof follows on the lines of Theorem 3.4. □

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